Abandonment of a hypothesis as to measurement of time from special theory of relativity

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In the special theory of relativity, an event in a four-dimensional space-time continuum is assigned three space co-ordinates and a time co-ordinate. In order to assign a time co-ordinate to an event, we fix the time of the event by means of clock. In fact, the fixing of time of an event by means of clock is based on the hypothesis that a physical process which constitutes a cycle of clock always fills the same length of time. This hypothesis can be stated more generally as follows, since a clock is a cyclic physical system: a physical process always fills the same length of time. Here, we show that this hypothesis can be abandoned from the special theory of relativity, and that the abandonment of the hypothesis retains the form of the theory. The abandonment of the hypothesis leads to a modification of the interpretation of the form of the special theory of relativity: Abusively speaking, the abandonment of the hypothesis removes the concept of time as a physical entity from the theory.

The hypothesis is conspicuously stated in Hermann Weyl's treatise[1]. We cite here his expression. "The empirical content which fills the length of Time AB can in itself be put into any other time without being in any way different from what it is. The length of time which it would then occupy is equal to the distance AB." This statement is a hypothesis. In fact, we have no means of proving the truth of this statement.

In the first half of the paper, we will show that the hypothesis that a physical process always fills the same length of time can be abandoned from the special theory of relativity. In the second half of the paper, we will show that the abandonment of this hypothesis retains the form of the theory. Throughout this paper, the concept of simultaneity is based on the definition which is given by Einstein. We consider only inertial systems and bodies which are in uniform straight motion relatively to inertial systems. We will call the observer which is placed at rest in the inertial system Φ at the origin simply the inertial observer Φ.

Our first task is to describe the motion of a body, relating to the motion of another body. Let Φ be an arbitrary inertial observer. Let i be an arbitrary body which is in uniform straight motion relatively to Φ. Let us denote the distance from the inertial observer Φ to the body i by Φ_i [m]. Then we describe Φ_i as a function of t [s] as follows: $\Phi_i = \phi_i(t)$, where t denotes the time parameter of Φ. Let us assume that the body i satisfies the condition $\phi_i(0) = 0 \cdots (*)$ which means that at the time $t = 0$ the inertial observer Φ and the body i coincide. Then the mapping $\phi_i : [0,\infty) \to [0,\infty); t \mapsto \Phi_i$ is a bijection, that is, the mapping ϕ_i is an one-to-one correspondence. Let j be also an arbitrary body which is in uniform straight motion relatively to Φ. We assume that the body j also satisfies the condition (*). Then we have $\phi_j(0) = 0$. Thus the mapping $\phi_j : [0,\infty) \to [0,\infty); t \mapsto \Phi_j$ is also bijection. Now, under these assumptions, we'd like to describe the quantity Φ_i as a function of the quantity Φ_j. Remember that we have the relations $\Phi_i = \phi_i(t)$, $\Phi_j = \phi_j(t)$. Since ϕ_j is a bijection, ϕ_j has the inverse mapping ϕ_j^{-1}. Thus we have the equation $\phi_j^{-1}(\Phi_j) = t$. Introduction of this equation into $\Phi_i = \phi_i(t)$ gives $\Phi_i = \phi_i(\phi_j^{-1}(\Phi_j))$. Therefore we obtain the equation $\Phi_i = (\phi_i \circ \phi_j^{-1})(\Phi_j)$. This equation enables us to describe Φ_i as a function of Φ_j. Moreover, since this equation expresses how the body i changes Φ_i with Φ_j, this equation enables us to describe the motion of i relating to the motion of j. For simplicity, throughout this paper we assume that any body which is in uniform straight motion relatively to the inertial observer Φ satisfies the condition (*). Note that then the speed of i relatively to Φ can be written as $\frac{d\Phi_i}{dt}$ [m/s], where i denotes an arbitrary body which is in uniform straight motion relatively to Φ.

Our next task is to define the concept of "generalized speed", which plays a crucial role in the argument below. In the above paragraph, we saw that we can describe Φ_i as a function of Φ_j by the equation $\Phi_i = (\phi_i \circ \phi_j^{-1})(\Phi_j)$. Now we define the quantity V_{ij} by $V_{ij} = \lim_{\Delta\Phi_j \to 0} \frac{\Delta\Phi_i}{\Delta\Phi_j} \left(= \lim_{\Delta\Phi_j \to 0} \frac{\Delta(\phi_i \circ \phi_j^{-1})(\Phi_j)}{\Delta\Phi_j} \right)$. This limit exists. Indeed, since the body i is in uniform straight motion relatively to Φ, the bijection ϕ_i is a C^1 diffeomorphism, that is, both ϕ_i and ϕ_i^{-1} are C^1 maps. Similarly ϕ_j is also a C^1 diffeomorphism. Then the inverse mapping ϕ_j^{-1} is also a C^1 diffeomorphism. Since both ϕ_i and ϕ_j^{-1} are C^1 diffeomorphisms, the composite mapping $\phi_i \circ \phi_j^{-1}$ is also a C^1 diffeomorphism. Thus we see that the above limit exists. Therefore V_{ij} can be written as $V_{ij} = \frac{d\Phi_i}{d\Phi_j}$. We define the following word.

Definition. Let i , j be two arbitrary bodies which are in uniform straight motion relatively to the inertial observer Φ. Then we will call the quantity V_{ij} "generalized speed" in order to make a distinction between V_{ij} and speed in the usual sense.

In this paper we call speed in the usual sense simply speed. Whereas speed is a ratio between the distance that a body moves and the time that the body takes to move, generalized speed is a ratio between the distance that a body moves and the distance that another body moves. Thus V_{ij} is a dimensionless quantity. Let us define the following words.

Definition. Let us consider the generalized speed $V_{ij} = \frac{d\Phi_i}{d\Phi_j}$. Then we will call the body j the "criterion body" of the generalized speed V_{ij}, and we will call the body i the "object body" of the generalized speed V_{ij}.

Generalized speed is a quantity which depends on an inertial observer, a criterion body and an object body. In particular, a generalized speed of i relatively to Φ depends on the choice of criterion body.

Let us record some elementary properties of generalized speed. We have the following natural proposition.

Proposition. Let v_i [m/s] be the speed of i relatively to Φ, v_j [m/s] the speed of j relatively to Φ. Then, it follows that $V_{ij} = \frac{d\Phi_i}{d\Phi_j} = \frac{v_i}{v_j}$ [dimensionless].

Proof. By the assumption, we have the relations $\Phi_i = \phi_i(t) = v_i t$, $\Phi_j = \phi_j(t) = v_j t$. (Remember that we assume that $\phi_i(0) = 0$, $\phi_j(0) = 0$.) Thus it follows that

$\Phi_i = \phi_i(t) = \phi_i(\phi_j^{-1}(\Phi_j)) = \phi_i(\frac{\Phi_j}{v_j}) = v_i \cdot \frac{\Phi_j}{v_j}$. Therefore we obtain

$$V_{ij} = \frac{d\Phi_i}{d\Phi_j} = (v_i \cdot \frac{\Phi_j}{v_j}) \cdot \frac{d}{d\Phi_j} = \frac{v_i}{v_j}.$$

In particular, we directly obtain the following corollary.

Corollary. For any i, we have $V_{ii} = 1$ [dimensionless].

If we visualize the equation in the above corollary in the xy-plane whose each axis corresponds to the quantity Φ_i, the straight line that represents the equation makes a 45-degree angle with the axes. This graph represents nothing else than the trivial statement that while the body i moves a [m] relatively to Φ, the body i moves a [m] relatively to Φ. We note the following:

Remark. If the inertial observer Φ chooses the body j as a criterion body of generalized speed, then Φ can't consider any substantial generalized speed of j; If Φ wishes to consider a generalized speed of j, Φ can consider only V_{jj}. However, this generalized speed is senseless as we saw in the corollary.

Now, the inertial observer Φ may regard a generalized speed $V_{ij} = \frac{d\Phi_i}{d\Phi_j}$ as a speed $v = \frac{d\Phi_i}{dt}$. Indeed, the inertial observer Φ can choose a criterion body in order that for any i, the generalized speed of i be numerically identical with the speed of i. Imagine, for example, an ideal body j which is in uniform straight motion with speed 1[m/s] relatively to Φ. Let Φ choose j as a criterion body. Then, for all i, the generalized speed of i is numerically identical with the speed of i. Indeed, if v [m/s] is the speed of i relatively to Φ, then the generalized speed of i $V_{ij} = \frac{d\Phi_i}{d\Phi_j}$ is given by $V_{ij} = \frac{d\Phi_i}{d\Phi_j} = \frac{v}{1} = v$ [dimensionless]. This directly follows from proposition. Thus the generalized speed of i $V_{ij} = \frac{d\Phi_i}{d\Phi_j}$ is numerically identical with the speed of i $v = \frac{d\Phi_i}{dt}$, where i denotes an arbitrary body which is in uniform straight motion relatively to Φ. Therefore we see that the inertial observer Φ may regard generalized speed whose criterion body is j as speed. A natural question arising from this consideration is whether the equivalence between $\frac{d\Phi_i}{d\Phi_j}$ and $\frac{d\Phi_i}{dt}$ indicates the equivalence as a parameter between Φ_j and t. We conclude that the parameter t is nothing but the parameter Φ_j. In other words, we entitle the inertial observer Φ to regard the passage of time as the motion of a criterion body. Then time, which is measured by clock, is discarded as a physical entity, and then the hypothesis that a physical process always fills the same length of time becomes superfluous. As a consequence of the abandonment of the hypothesis, we are forced to discard the concept of speed, which contains the concept of time. The concept of generalized speed will replace the concept of speed. We note that the concept of simultaneity and clock as the apparatus to determine whether two arbitrary events took place simultaneously or not are still needed.

From now on, we will develop a similar argument considering two inertial observers in order to show that the abandonment of the hypothesis retains the form of the special theory of relativity. Let Φ, Ψ be two arbitrary inertial observers. We denote the time parameter of Φ by t_Φ [s], and we denote the time parameter of Ψ by t_Ψ [s]. Throughout this paper, for simplicity, we assume that the inertial observer Ψ coincides with the inertial observer Φ at the time $t_\Phi = t_\Psi = 0$. Then we have $\phi_\Psi(0) = 0$, $\psi_\Phi(0) = 0$. Thus we also have $\psi_i(0) = 0$, where i denotes an arbitrary body which is in uniform straight motion relatively to Φ. (remember that we assume $\phi_i(0) = 0$) Note that then the speed of i relatively to Φ can be written as $\frac{d\Phi_i}{dt_\Phi}$ [m/s], and that then the speed of i relatively to Ψ can be written as $\frac{d\Psi_i}{dt_\Psi}$ [m/s], where i denotes an arbitrary body which is in uniform straight motion relatively to Φ. For example, the speed of Ψ relatively to Φ can be written as $\frac{d\Phi_\Psi}{dt_\Phi}$, and the speed of Φ relatively to Ψ can be written as $\frac{d\Psi_\Phi}{dt_\Psi}$.

We first give our attention to a special criterion body. We define the following word.

Definition. We will call a criterion body of generalized speed which satisfies the following condition (**) “absolute criterion”.

(**): If both Φ and Ψ choose an absolute criterion as a criterion body, then the generalized speed of Φ relatively to Ψ is numerically identical with the generalized speed of Ψ relatively to Φ. In other words, if α is an absolute criterion, it follows that

$$\frac{d\Psi_\Phi}{d\Psi_\alpha} = \frac{d\Phi_\Psi}{d\Phi_\alpha}.$$

We obtain the following theorem.

Theorem1. A ray of light is an absolute criterion.

Proof. We denote a ray of light by α. We assume that α satisfies the condition (*). Then we have $\phi_\alpha(0) = 0$. Further, it also follows that $\psi_\alpha(0) = 0$, for we have $\phi_\Psi(0) = 0$. In other words, we assume that at the time $t_\Phi = t_\Psi = 0$ the inertial observer Φ, the inertial observer Ψ and the emitting body of the ray of light α coincide, and that α is emitted at the time $t_\Phi = t_\Psi = 0$. We now assume that both Φ and Ψ choose α as a criterion body of generalized speed. Let v [m/s] be the speed of Ψ relatively to Φ. Then v [m/s] is also the speed of Φ relatively to Ψ. Under these assumptions, the generalized speed of Ψ relatively to Φ $V_{\Psi\alpha} = \frac{d\Phi_\Psi}{d\Phi_\alpha}$ is given by $V_{\Psi\alpha} = \frac{d\Phi_\Psi}{d\Phi_\alpha} = \frac{v}{c}$ [dimensionless], where c [m/s] denotes the speed of light. To prove the theorem, we must show that the generalized speed of Φ relatively to Ψ $W_{\Phi\alpha} = \frac{d\Psi_\Phi}{d\Psi_\alpha}$ (here we set $W_{\Phi\alpha} = \frac{d\Psi_\Phi}{d\Psi_\alpha}$) is also given by $W_{\Phi\alpha} = \frac{d\Psi_\Phi}{d\Psi_\alpha} = \frac{v}{c}$ [dimensionless]. To prove this, we must show that Φ moves v [m] relatively to Ψ while α moves c [m] relatively to Ψ. Thus we now assume that α moves c [m] relatively to Ψ. Then as judged from Ψ, the time 1[s] elapses, because c [m/s] is the speed of α relatively to Ψ. Since Φ is in uniform straight motion with speed v [m/s] relatively to Ψ, Φ moves v [m] relatively to Ψ in this time interval 1[s]. Therefore the generalized speed of Φ relatively to Ψ

$W_{\Phi\alpha} = \dfrac{d\Psi_{\Phi}}{d\Psi_{\alpha}}$ is given by $W_{\Phi\alpha} = \dfrac{d\Psi_{\Phi}}{d\Psi_{\alpha}} = \dfrac{v}{c}$ [dimensionless]. Thus we see that the ray of light α is an absolute criterion, for we now have $\dfrac{d\Psi_{\Phi}}{d\Psi_{\alpha}} = \dfrac{d\Phi_{\Psi}}{d\Phi_{\alpha}} = \dfrac{v}{c}$.

Since light is the only object whose speed is independent of the states of motion of the observers, we also see that only light can be absolute criterion. Thus we obtain the following theorem.

Theorem2. In order that a criterion body be an absolute criterion it is necessary and sufficient that the criterion body is a ray of light.

Absolute criterion is criterion body common to Φ and Ψ. Let us define the word that specifies an observer of absolute criterion.

Definition. Let a ray of light α be an absolute criterion. Then we will call the absolute criterion α that is observed by the inertial observer Φ the "proper criterion of Φ", and we will call the absolute criterion α that is observed by the inertial observer Ψ the "proper criterion of Ψ". We denote the proper criterion of Φ by (α, Φ), and we denote the proper criterion of Ψ by (α, Ψ).

When we don't specify an observer of the absolute criterion α, we will call the absolute criterion α simply the absolute criterion α and denote it by simply α.

Our task is now to show that generalized speed whose criterion body is absolute criterion is equivalent to speed for Φ and for Ψ. We have the following theorem.

Theorem3. Let a ray of light α be an absolute criterion. Then, for the inertial observer Φ and for the inertial observer Ψ, generalized speed whose criterion body is α is equivalent to speed.

Let us formulate the above assertion mathematically. It can be stated as follows:

The concept $\frac{d\Phi_i(\Phi_\alpha)}{d\Phi_\alpha}$ (generalized speed) is equivalent to the concept $\frac{d\Phi_i(t_\Phi)}{dt_\Phi}$ (speed), and the concept $\frac{d\Psi_i(\Psi_\alpha)}{d\Psi_\alpha}$ (generalized speed) is equivalent to the concept $\frac{d\Psi_i(t_\Psi)}{dt_\Psi}$ (speed), where i denotes an arbitrary body which is in uniform straight motion relatively to Φ and Ψ.

Proof. Let us prove that generalized speed whose criterion body is α satisfies the following three properties:

(1) Symmetry: The generalized speed of Φ relatively to Ψ $W_{\Phi\alpha} = \frac{d\Psi_\Phi}{d\Psi_\alpha}$ is numerically identical with the generalized speed of Ψ relatively to Φ $V_{\Psi\alpha} = \frac{d\Phi_\Psi}{d\Phi_\alpha}$.

(2) Constancy of the generalized speed of light: The generalized speed of β relatively to Φ $V_{\beta\alpha} = \frac{d\Phi_\beta}{d\Phi_\alpha}$ and the generalized speed of β relatively to Ψ $W_{\beta\alpha} = \frac{d\Psi_\beta}{d\Psi_\alpha}$ are a determined constant, where β denotes a ray of light which is emitted from an arbitrary body which is in uniform straight motion relatively to Φ and Ψ. (we assume that $\phi_\beta(0) = 0$, $\psi_\beta(0) = 0$)

(3) Numerical value: The generalized speed of i relatively to Φ $V_{i\alpha} = \frac{d\Phi_i}{d\Phi_\alpha}$ is numerically identical with the speed of i relatively to Φ $v = \frac{d\Phi_i}{dt_\Phi}$ (here we set $v = \frac{d\Phi_i}{dt_\Phi}$), and the generalized speed of i relatively to Ψ $W_{i\alpha} = \frac{d\Psi_i}{d\Psi_\alpha}$ is numerically identical with the speed of i relatively to Ψ $w = \frac{d\Psi_i}{dt_\Psi}$ (here we set $w = \frac{d\Psi_i}{dt_\Psi}$), where i denotes an arbitrary body which is in uniform straight motion relatively to Φ and Ψ.

First, let us prove that generalized speed whose criterion body is α satisfies the property (1). This directly follows from the definition of absolute criterion and theorem1. From theorem2, we also see that generalized speed whose criterion body is not a ray of light doesn't satisfy this property. Next, let us prove that generalized speed whose criterion body is α satisfies the property (2). To do this, let us calculate the generalized speed of β relatively to Φ $V_{\beta\alpha} = \frac{d\Phi_\beta}{d\Phi_\alpha}$. While α moves c [m] relatively to Φ, as judged from Φ, the time 1[s] elapses, because c [m/s] is the speed of α relatively to Φ. Then, by the principle of the constancy of the speed of light, β also moves c [m] relatively to Φ in this time interval 1[s]. Thus the generalized speed $V_{\beta\alpha} = \frac{d\Phi_\beta}{d\Phi_\alpha}$ is given by $V_{\beta\alpha} = \frac{d\Phi_\beta}{d\Phi_\alpha} = \frac{c}{c} = 1$ [dimensionless]. Similarly, we obtain also $W_{\beta\alpha} = \frac{d\Psi_\beta}{d\Psi_\alpha} = 1$ [dimensionless]. Therefore we obtain $V_{\beta\alpha} = W_{\beta\alpha} = 1$. This formula shows that the generalized speed of a ray of light is a determined constant 1. Finally, let us prove that generalized speed whose criterion body is α satisfies the property (3). Let v [m/s] be the speed of i relatively to Φ. Then the generalized speed of i relatively to Φ $V_{i\alpha} = \frac{d\Phi_i}{d\Phi_\alpha}$ is given by $V_{i\alpha} = \frac{d\Phi_i}{d\Phi_\alpha} = \frac{v}{c}$ [dimensionless]. Thus there exists an apparent incompatibility of the generalized speed of i with the speed of i. However, it is not a fundamental problem how we define a time interval 1[s]. If we define a time interval 1[s] as the time that light needs to move 1[m], the speed of i relatively to Φ is given by $\frac{d\Phi_i}{dt_\Phi} = \frac{v}{c}$ [m/s]. Then the generalized speed of i relatively to Φ is numerically identical with the speed of i relatively to Φ. From the foregoing three proofs, we conclude that the theorem3 was proved.

The theorem3 states that the inertial observer Φ may regard generalized speed whose criterion body is proper criterion of Φ as speed, and that the inertial observer Ψ may regard generalized speed whose criterion body is proper criterion of Ψ as speed.

The previous theorem leads us to the main result of the second half of the paper. We again give our attention to their parameters.

Theorem4. Let a ray of light α be an absolute criterion. Then, the parameter Φ_α [m] is equivalent to the parameter t_Φ [s], and the parameter Ψ_α [m] is equivalent to the parameter t_Ψ [s]. In other words, the proper criterion of Φ (α, Φ) is equivalent to the proper time of Φ, and the proper criterion of Ψ (α, Ψ) is equivalent to the proper time of Ψ.

The equivalence as a parameter between $\Phi_\alpha\ (\Psi_\alpha)$ and $t_\Phi\ (t_\Psi)$ entitles the inertial observer Φ (Ψ) to regard the passage of time as the propagation of the ray of light α. In the special theory of relativity, a unit of time is chosen so that the speed of light becomes equal to unity. To choose a unit of time so that the speed of light becomes equal to unity is to choose a ray of light as criterion body, which means that our assertion that the passage of time is the motion of criterion body is formally recorded in the special theory of relativity. Therefore we see that the abandonment of the hypothesis retains the form of the special theory of relativity.

1. Weyl, H. *Space-Time-Matter* (Dover, New York, 1952)